

THE DIRICHLET PROBLEM FOR THE SLAB WITH ENTIRE DATA AND A DIFFERENCE EQUATION FOR HARMONIC FUNCTIONS

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ABSTRACT. It is shown that the Dirichlet problem for the slab $(a, b) \times \mathbb{R}^d$ with entire boundary data has an entire solution. The proof is based on a generalized Schwarz reflection principle. Moreover, it is shown that for a given entire harmonic function g the inhomogeneous difference equation $h(t+1, y) - h(t, y) = g(t, y)$ has an entire harmonic solution h .

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1. INTRODUCTION

It is well known that the Dirichlet problem for *unbounded* domains differs in many respects from the case of bounded domains due to the non-uniqueness of solutions. An excellent discussion of the Dirichlet problem for general unbounded domains can be found in [10].

Maybe the simplest example of this kind is the Dirichlet problem for the strip $(a, b) \times \mathbb{R}$ which has been considered by Widder in [23], see also [6]. A discussion of the Dirichlet problem for half-spaces can be found in [9], [21], and for a cylinder in [19].

In this paper we are concerned with the harmonic extendibility of the solution of the Dirichlet problem for entire data on the slab (see [5])

$$S_{a,b} := (a, b) \times \mathbb{R}^d.$$

We say that a function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is entire if there exists an analytic function $F : \mathbb{C}^d \rightarrow \mathbb{C}$ such that $F(x) = f(x)$ for all $x \in \mathbb{R}^d$. Thus, an entire function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is real analytic, and it possesses an everywhere convergent power series expansion. It is well known that every harmonic function $h : \mathbb{R}^d \rightarrow \mathbb{C}$ is entire.

Our first main result in this paper is the following:

Theorem 1. *Let h be a solution of the Dirichlet problem for the slab $S_{a,b}$ for entire data $f_0, f_1 : \mathbb{R}^d \rightarrow \mathbb{C}$, i.e. h is harmonic on S and $\lim_{t \rightarrow a} h(t, y) = f_0(y)$ and $\lim_{t \rightarrow b} h(t, y) = f_1(y)$. Then h extends to all of \mathbb{R}^{d+1} as a harmonic function.*

A similar result holds for the Dirichlet problem for the ellipsoid: H.S. Shapiro and the first author have established in [17] that for each entire data function there exists a solution of the Dirichlet problem which extends to a harmonic function defined on \mathbb{R}^d , see also [3] for further extensions. For the case of a cylinder with ellipsoidal base it is not yet known whether for any entire data function there exists an entire harmonic solution,

see [18] and [11] for partial results. We refer the reader to a discussion in [7], [16], [13] and [20] regarding the question of which domains Ω allow entire extensions for entire data.

From Theorem 1 we shall derive our second main result:

Theorem 2. *If $g : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ is harmonic then the difference equation*

$$(1) \quad h(t+1, y) - h(t, y) = g(t, y)$$

has a harmonic solution $h : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$.

Let us recall now some notations and definitions. A function $f : \Omega \rightarrow \mathbb{C}$ defined on a domain Ω in the Euclidean space \mathbb{R}^d is called *harmonic* if f is twice continuously differentiable and $\Delta f(x) = 0$ for all $x \in \Omega$ where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}$$

is the Laplace operator. We also write Δ_x instead of Δ to indicate the variables for differentiation. We say that a function $g : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ is even (odd, respectively) at t_0 if

$$g(t_0 + t, y) = g(t_0 - t, y),$$

and $g(t_0 + t, y) = -g(t_0 - t, y)$, respectively, for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^d$.

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2. THE DIRICHLET PROBLEM ON THE SLAB WITH ENTIRE DATA

Suppose $h : [a, b] \times \mathbb{R}^d \rightarrow \mathbb{C}$ is continuous and harmonic in the open slab $(a, b) \times \mathbb{R}^d$ such that $h(a, y) = h(b, y) = 0$ for all $y \in \mathbb{R}^d$. Then it is a well known consequence of the Schwarz reflection principle that h extends to a harmonic function on \mathbb{R}^{d+1} which is periodic in the variable t with period $2(b - a)$, i.e.

$$h(t + 2(b - a), y) = h(t, y).$$

In order to obtain a similar result with arbitrary entire boundary data, we shall need the following extension of the Schwarz reflection principle:

Theorem 3. *Suppose that Ω is a domain in \mathbb{R}^{d+1} such that for each $x = (x_1, \dots, x_{d+1}) \in \Omega$ the vector $\tilde{x} = (-x_1, x_2, \dots, x_{d+1}) \in \Omega$, and let $\Omega_+, \Omega_0, \Omega_-$ denote the sets of points $x \in \Omega$ for which x_1 is positive, zero, and negative (respectively). Suppose that $y \mapsto F(y)$ for $y = (x_2, \dots, x_{d+1}) \in \mathbb{R}^d$ is an entire function, and assume that h is harmonic on Ω_- such that for all $y \in \Omega_0$ we have $h(x) \rightarrow F(y)$ as $x \rightarrow y \in \Omega_0$. Then h has a harmonic extension to Ω .*

Proof. By the Cauchy-Kovalevskaya Theorem applied to the Laplace operator (see [14, p. 80, Example 11.2]), there is a unique entire function H such that $H(0, y) = F(y)$ and $\frac{\partial}{\partial x}H(0, y) = 0$ for all $y \in \mathbb{R}^d$. Moreover, from the uniqueness part of the Cauchy-Kovalevskaya Theorem it follows that H is even at $t_0 = 0$, since $H(-t, y)$ solves the same Cauchy problem as $H(t, y)$. Consider the function

$$f(t, y) := h(t, y) - H(t, y)$$

for $(t, y) \in \Omega_-$. Then for each $y \in \mathbb{R}^d$ we have $f(t, y) \rightarrow 0$ as $t \rightarrow 0$, and by the Schwarz reflection principle (see [2, p. 8]) f extends to a harmonic function \tilde{f} on Ω by the formula

$$\tilde{f}(t, y) = -f(-t, y)$$

for all $(t, y) \in \Omega_+$. Then

$$\tilde{h}(t, y) := \tilde{f}(t, y) + H(t, y)$$

is a harmonic extension of h from Ω_- to Ω , and for $t > 0$ we have

$$(2) \quad \tilde{h}(t, y) = \tilde{f}(t, y) + H(t, y) = -f(-t, y) + H(t, y) = -h(-t, y) + 2H(t, y).$$

□

Now we obtain our first main result:

Theorem 4. *Assume that $h \in C([a, b] \times \mathbb{R}^d)$ is harmonic in the slab $(a, b) \times \mathbb{R}^d$ such that $y \mapsto h(a, y)$ and $y \mapsto h(b, y)$ are entire. Then there exists a harmonic extension $\tilde{h} : \mathbb{R}^{d+1} \rightarrow \mathbb{C}$.*

Proof. The following provides an inductive step:

Claim. There is an extension $\tilde{h} \in C([a, 2b - a] \times \mathbb{R}^d)$ of h which is harmonic in $(a, 2b - a) \times \mathbb{R}^d$ such that $y \mapsto \tilde{h}(2b - a, y)$ is entire.

Using the Claim and induction, one obtains a harmonic extension on $(a, b + n(b - a)) \times \mathbb{R}^d$ for each natural number n such that $y \mapsto \tilde{h}(a + n(b - a), y)$ is entire. Similarly, there is a harmonic extension on $(a - n(b - a), b) \times \mathbb{R}^d$ for each natural number n , and the proof is complete.

In order to establish the Claim, we may assume that $a < b = 0$. Theorem 3 provides an extension $\tilde{h}(t, y)$ of $h(t, y)$ to $[a, -a] \times \mathbb{R}^d$. Moreover, from the proof of Theorem 3, $\tilde{h}(t, y)$ is given by Equation (2)

$$\tilde{h}(t, y) = -h(-t, y) + 2H(t, y),$$

where H is an entire harmonic function. This implies that the restriction $\tilde{h}(-a, y)$ is entire since $h(a, y)$ is assumed entire, and the result then follows. □

Corollary 5. *Let $f_0, f_1 : \mathbb{R}^d \rightarrow \mathbb{C}$ be entire functions. Then any solution h for the Dirichlet problem for the slab $(a, b) \times \mathbb{R}^d$ with $h(a, y) = f_0(y)$ and $h(b, y) = f_1(y)$ extends to a harmonic function $h : \mathbb{R}^{d+1} \rightarrow \mathbb{C}$.*

Proof. It is known that the Dirichlet problem for the slab with continuous data has a solution h , see e.g. [10]. By Theorem 4 the function h has an entire extension. \square

3. THE DIFFERENCE EQUATION FOR HARMONIC FUNCTIONS

We recall from complex analysis [4, p. 407] that the inhomogeneous difference equation

$$(3) \quad f(z+1) - f(z) = G(z)$$

for a given entire function $G(z)$ has an entire solution $f(z)$. This is a classical fact, and the solution given in [4, p. 407] uses Bernoulli polynomials, an idea which goes back to the work of Guichard, Appel, and Hurwitz more than a century ago (see [1, 12]).

Taking the real part of both sides and recalling that any harmonic function $g(t, y)$ in the plane is the real part $\Re \{G(t+iy)\}$ of some entire function $G(z)$, it follows that the difference equation

$$(4) \quad h(t+1, y) - h(t, y) = g(t, y)$$

for a given harmonic function g on \mathbb{R}^2 has a harmonic solution h defined on \mathbb{R}^2 .

In this section, we shall generalize this result to all dimensions of the variable $y \in \mathbb{R}^d$. Our approach is based on solving the Dirichlet problem for the slab $[0, 1/2] \times \mathbb{R}^d$, and thus we do not need special functions as in the above-mentioned classical studies.

It is a remarkable fact that equation (3) can be solved for meromorphic functions as well. It would be interesting to extend our results to include the difference equation (4) for g with singularities (say of a controlled type).

As an intermediate step toward solving the difference equation (4), we provide a solution in the case when $g(t, y)$ is even:

Theorem 6. *Let $g : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ be harmonic and even. Then any solution $h(t, y)$ of the Dirichlet problem for the slab $[0, 1/2] \times \mathbb{R}^d$ with data*

$$h(0, y) = -\frac{1}{2}g(0, y) \text{ and } h\left(\frac{1}{2}, y\right) = 0$$

for all $y \in \mathbb{R}^d$ induces an entire harmonic solution of the difference equation

$$h(t+1, y) - h(t, y) = g(t, y).$$

Proof. By Corollary 5 there exists an entire harmonic function $h(t, y)$ such that $h(0, y) = -\frac{1}{2}g(0, y)$ and $h(\frac{1}{2}, y) = 0$. The last equation and the Schwarz reflection principle shows that

$$(5) \quad h\left(\frac{1}{2} + t, y\right) = -h\left(\frac{1}{2} - t, y\right).$$

Inserting $t = \frac{1}{2}$ in equation (5) gives $h(1, y) = -h(0, y) = \frac{1}{2}g(0, y)$. Now we consider the harmonic function

$$F(t, y) = h(t, y) - \frac{1}{2}g(t-1, y).$$

Then $F(1, y) = 0$, and by Schwarz's reflection principle, $F(1 + t, y) = -F(1 - t, y)$ for $y \in \mathbb{R}^d$. Then

$$\begin{aligned} h(1 + t, y) &= F(1 + t, y) + \frac{1}{2}g(t, y) = -F(1 - t, y) + \frac{1}{2}g(t, y) \\ &= -h(1 - t, y) + \frac{1}{2}g(-t, y) + \frac{1}{2}g(t, y) \end{aligned}$$

Since g is even we have $\frac{1}{2}g(-t, y) + \frac{1}{2}g(t, y) = g(t, y)$ and

$$h(1 - t, y) = h\left(\frac{1}{2} + \frac{1}{2} - t, y\right) = -h\left(\frac{1}{2} - \left(\frac{1}{2} - t\right), y\right) = -h(t, y).$$

It follows that $h(1 + t, y) = h(t, y) + g(t, y)$. \square

The next result is surely a part of mathematical folklore; we include an elementary proof in order to keep the paper self-contained.

Lemma 7. *Let $g(t, y)$ be an entire harmonic function. Then there exists an entire harmonic function $u(t, y)$ such that*

$$\frac{\partial}{\partial t}u(t, y) = g(t, y).$$

If $g(t, y)$ is odd then $u(t, y)$ can be chosen to be even.

Proof. Define $h(t, y) := \int_0^t g(\tau, y) d\tau$. Then $\frac{\partial}{\partial t}h(t, y) = g(t, y)$ and

$$\Delta_y \frac{\partial}{\partial t}h(t, y) = \Delta_y g(t, y) = -\frac{\partial^2}{\partial t^2}g(t, y).$$

We conclude that $\frac{\partial}{\partial t}(\Delta_y h(t, y) + \frac{\partial}{\partial t}g(t, y)) = 0$, and it follows that

$$f(y) := \Delta_y h(t, y) + \frac{\partial}{\partial t}g(t, y)$$

only depends on y and not on t . Obviously for any entire function $f(y)$ there exists an entire function $G(y)$ such that $\Delta_y G(y) = f(y)$. Then

$$u(t, y) := h(t, y) - G(y)$$

is a solution of the equation $\frac{\partial}{\partial t}u(t, y) = g(t, y)$, and $u(t, y)$ is harmonic, since we have

$$\begin{aligned} \Delta_{t,y}u(t, y) &= \frac{\partial^2}{\partial t^2}h(t, y) + \Delta_y h(t, y) - \Delta_y G(y) \\ &= \frac{\partial^2}{\partial t^2}h(t, y) + \Delta_y h(t, y) - \Delta_y h(t, y) - \frac{\partial}{\partial t}g(t, y) = 0. \end{aligned}$$

If $g(t, y)$ is odd, then $h(t, y)$ and hence $u(t, y)$ are both even. \square

Now we are able to prove our second main result:

Theorem 8. *If $g : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ is harmonic then the difference equation $h(t + 1, y) - h(t, y) = g(t, y)$ has a harmonic solution $h : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$.*

Proof. Write the harmonic function g as a sum $g_0 + g_e$, where g_0 is odd and g_e is even. For right hand side g_e , there exists a solution $h_e(t, y)$ by Theorem 6, and so it suffices to solve the difference equation with right hand side g_0 . By Lemma 7 there exists an even harmonic function $u(t, y)$ such that

$$\frac{\partial}{\partial t} u(t, y) = g_0(t, y).$$

By Theorem 6, there exists a harmonic entire function $H(t, y)$ such that

$$H(t+1, y) - H(t, y) = u(t, y).$$

Differentiating $H(t, y)$ with respect to t , we thus find the solution of the difference equation with right hand side $g_0(t, y)$. \square

Finally, although the solution to the difference equation is far from unique, we are able to prove the following result.

Theorem 9. *Let $g : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a harmonic function and let $h_j : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ for $j = 1, 2$ be harmonic solutions of the difference equation*

$$(6) \quad h_j(t+1, y) - h_j(t, y) = g(t, y).$$

Assume that we have the estimate

$$(7) \quad |h_1(t, y) - h_2(t, y)| = o\left(|y|^{(1-d)/2} e^{2\pi|y|}\right),$$

as $|y| \rightarrow \infty$ (uniformly in t). Then there exists a harmonic function $r : \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$h_1(t, y) = h_2(t, y) + r(y)$$

for all $y \in \mathbb{R}^d$ and $t \in \mathbb{R}$.

Proof. Define $h(t, y) := h_1(t, y) - h_2(t, y)$. Then h is periodic in t with period 1, since h_1 and h_2 each solve the difference equation (6). Let us also define $H(t, y) := H_y(t) := h(t/(2\pi), y/(2\pi))$ for $y \in \mathbb{R}^d$, $t \in \mathbb{R}$. Then H_y is a 2π -periodic function in t , so it has a Fourier series $\sum_{k=-\infty}^{\infty} a_k(y) e^{ikt}$. Applying the Laplace operator to the Fourier coefficients

$$(8) \quad a_k(y) = \frac{1}{2\pi} \int_0^{2\pi} H_y(t) e^{ikt} dt,$$

we have

$$\Delta_y a_k(y) = \frac{1}{2\pi} \int_0^{2\pi} \Delta_y H(t, y) e^{ikt} dt.$$

Since $H(t, y)$ is harmonic we know that

$$\Delta_y H(t, y) = -\frac{\partial^2}{\partial t^2} H(t, y).$$

Integration by parts then shows that

$$\Delta_y a_k(y) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 H}{\partial t^2}(t, y) \cdot e^{ikt} dt = k^2 a_k(y).$$

Hence $a_k(y)$ is a solution of the Helmholtz equation $\Delta_y a_k = k^2 a_k$. In view of (8) the estimate (7) carries over to $a_k(y)$. Since $k^2 > 0$, a classical result [8, p. 228], going back to work of I. Vekua and F. Rellich in the 1940's, yields that $a_k = 0$ for all $k \neq 0$. \square

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